# Problem Set
## (Area, Distance, and the Definite Integral)

1. Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using 3 rectangles (same width) and the right endpoints. Then improve your estimate using 6 rectangles (same width) and the right endpoints.

2. A car traveling for $5$ hours has a velocity given by
$$v(t) = \begin{cases} 40 \text{ if } 0 \leq t \leq 2 \\ 50 \text{ if } 2 < t \leq 3 \\ 20 + 5t \text{ if } 3 < t \leq 5 \end{cases}$$
What is the total distance traveled?

3. The speed of a runner increased steadily during the 1st 3 seconds of a race. Her speed at half-second intervals is given in the table. Find the lower and upper estimates for the distance that she traveled during these 3 seconds.

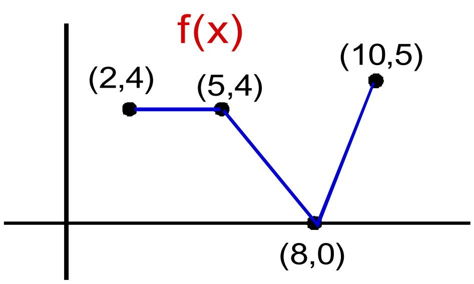
| $t$ in seconds | $0$ | $0.5$ | $1.0$ | $1.5$ | $2$ | $2.5$ | $3.0$ |
|---|---|---|---|---|---|---|---|
| $v$ in ft/sec | $0$ | $6.2$ | $10.8$ | $14.9$ | $18.1$ | $19.4$ | $20.2$ |

4. Let $f(x) = x^3 + 4$ where $-1 \leq x \leq 6$. Consider the partition $[-1, 2]$, $[2, 4]$, and $[4, 6]$. Find the Riemann Sum where $x_1$, $x_2$, and $x_3$ are
a) $-1$, $2$, and 4 b) 2, 3, and 4 c) 0, 4, and 6

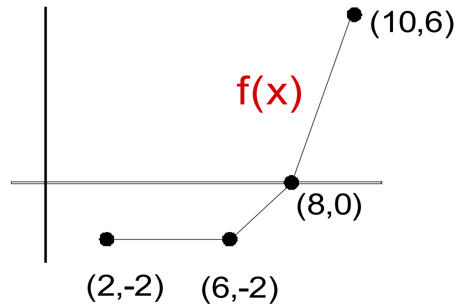
5. If $f(x) = \sqrt{x}$ where $4 \leq x \leq 16$. Consider the partition [4,6], [6,8], [8,10], [10,12], [12,14], and [14,16].
Find the Riemann Sum using the
a) right endpoints b) left endpoints c) the midpoints

6. Evaluate $\int_2^{10} f(x)dx$ given that $f(x)$ has the graph shown.

a)




b)




7. Use the midpoint rule to estimate the value of $\int_2^{10} \sqrt{x^3+1}dx$ using 4 rectangles of the same width. Evaluate this integral using your calculator.

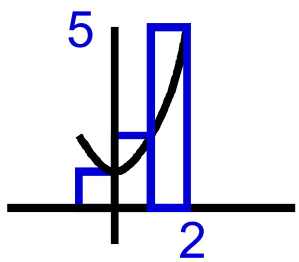
8. Use the midpoint rule to estimate the value of $\int_0^{3} e^{x/2}dx$ using 6 rectangles of the same width. Evaluate this integral using your calculator.

9. Evaluate the definite integral $\int_0^{10} \sqrt{100-x^2}\,dx$ by evaluating the area under the graph.

10. Evaluate the integral $\int_{-2}^{2} sin\,x\,dx$ by examining the area between the x-axis and the graph.

## Solutions

1.

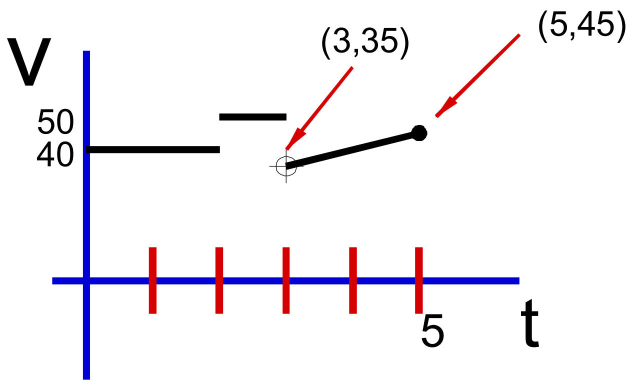



If we use 3 rectangles, then $[-1,2]$ is partitioned into $[-1,0]$, $[0,1]$, and $[1,2]$ where each rectangle has width $1$.
If we use the right endpoints, then the rectangles have height $1+0^2=1$, $1+1^2=2$, and $1+2^2=5$. The sum of the areas of the 3 rectangles is $(1)(1)+(1)(2)+(1)(5)=8$ which is a rough approximation of the area.

If we use 6 rectangles, then the partition of $[-1,2]$ is $[-1,-0.5]$, $[-0.5,0]$, $\ldots$, $[1.5,2]$. The rectangles have height $1+(-.5)^2$, $1+0^2$, $\ldots$, $1+(2)^2$. The area of the 6 rectangles is

$$\left[1+(-.5)^2\right](.5)+\left(1+0^2\right)(.5)+\left(1+.5^2\right)(.5)+\left(1+1^2\right)(.5)+\left(1+1.5^2\right)(.5)+\left(1+2^2\right)(.5)$$
$$=6.875 .$$

2. The graph of $v$ as a function of $t$ is shown below.




The total distance is the area between the t-axis and the graph. The total area is $(2)(40)+(1)(50)+(2)(40)=210$ and the total distance traveled is 210 miles. Note for the last 2 hours I used the average of 35 and 45 which is 40 mph.

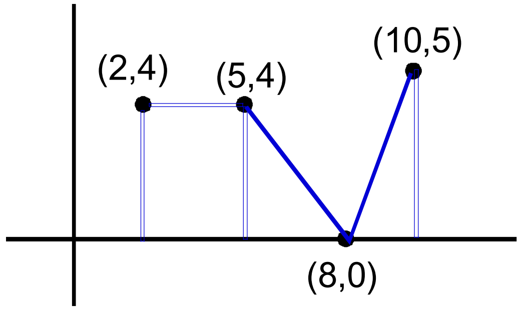
3. The lower estimate is obtained by using $v$ at the left endpoints of the subintervals. So for example, we assume that for the 1st 1/2 second $v = 0$, and for the next 1/2 second, $v = 6.2$.
Therefore the lower estimate is
$(.5)(0) + (.5)(6.2) + (.5)(10.8) + \ldots + (.5)(19.4) =$
$.5(0 + 6.2 + \cdots + 19.4) = 34.7$ feet.

The upper estimate uses $v$ at the right endpoints which means for the 1st 1/2 second we assume $v = 6.2$.
The upper estimate is
$.5\,(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) = 44.8$ ∎

4. a) The Riemann sum is
$$(3)\big((-1)^3 + 4\big) + (2)(2^3 + 4) + (2)(4^3 + 4) =$$
$$(3)(3) + (2)(12) + (2)(68) = 169$$
b) The Riemann sum is
$$(3)(2^3 + 4) + (2)(3^3 + 4) + (2)(4^3 + 4) =$$
$$(3)(12) + (2)(31) + (2)(68) = 234$$
c) The Riemann sum is
$$(3)(0 + 4) + (2)(4^3 + 4) + (2)(6^3 + 4)$$
$$(3)(4) + (2)(68) + (2)(220) = 588$$

5. a) Using the right endpoints the Riemann sum is
$$(2)\big(\sqrt{6}\big) + (2)\big(\sqrt{8}\big) + \ldots + (2)\big(\sqrt{16}\big) =$$
$$2\left(\sqrt{6} + \sqrt{8} + \sqrt{10} + \sqrt{12} + \sqrt{14} + \sqrt{16}\right) = 39.292 \blacksquare$$
b) Using the left endpoints the Riemann sum is
$$2\sqrt{4} + 2\sqrt{6} + \ldots + 2\sqrt{14} =$$
$$2\left(\sqrt{4} + \sqrt{6} + \sqrt{8} + \sqrt{10} + \sqrt{12} + \sqrt{14}\right) = 35.292 \blacksquare$$
c) Using the midpoints the Riemann sum is
$$2\sqrt{5} + 2\sqrt{7} + \ldots + 2\sqrt{15} =$$
$$2\left(\sqrt{5} + \sqrt{7} + \sqrt{9} + \sqrt{11} + \sqrt{13} + \sqrt{15}\right) = 37.354 \blacksquare$$

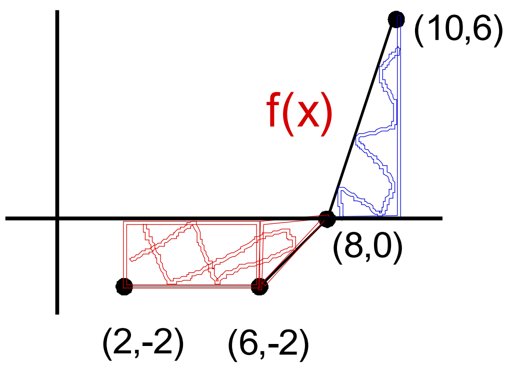
6. a)




The integral is equal to the sum of the areas of the rectangle and the 2 triangles. This total area is

$$(3)(4) + \frac{1}{2}(3)(4) + \frac{1}{2}(2)(5) = 12 + 6 + 5 = 23\,.$$

b)




The integral is equal to the area above the x-axis (blue area) minus the area below the x-axis (red area). This is equal to

$$\frac{1}{2}(2)(6) - \left(4 \times 2 + \frac{1}{2}2 \times 2\right) = 6 - 10 = -4\,.$$

7. If we use 4 rectangles of the same width then the partition is [2,4], [4,6], [6,8], and [8,10].

For most functions, it is best to use the midpoints of the subintervals. This is called the midpoint rule.

$$\int_2^{10} \sqrt{x^3+1}\,dx \approx (2)\left(\sqrt{3^3+1}\right) + (2)\left(\sqrt{5^3+1}\right) + (2)\left(\sqrt{7^3+1}\right) + (2)\left(\sqrt{9^3+1}\right) = 124.164$$

Using a calculator, $\int_2^{10} \sqrt{x^3+1}\, dx = 124.616$.

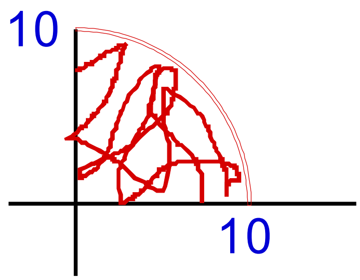
8. The subintervals are $[0,\ 0.5]$, $[0.5,\ 1]$, $\ldots$, $[2.5,\ 3]$.
The midpoints are 0.25, 0.75, 1.25, 1.75, 2.25, and 2.75 .

$$\int_0^3 e^{x/2} dx =$$

$$0.5\left(e^{.25/2} + e^{.75/2} + e^{1.25/2} + e^{1.75/2} + e^{2.25/2} + e^{2.75/2}\right) =$$

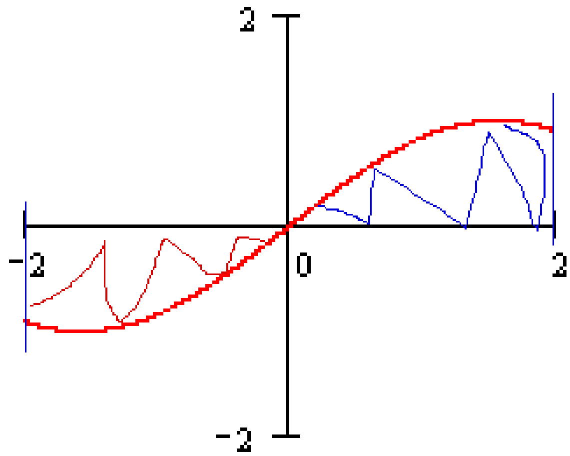
$6.945$. Using a calculator, $\int_0^3 e^{x/2} dx =$ 6.963 .

9. The graph of $y = \sqrt{100 - x^2}$ , $0 \le x \le 10$, is 1/4 of a circle.




The integral is equal to the area which is $\frac{1}{4}\pi r^2 = \frac{1}{4}\pi (10)^2 = 25\pi$.

10. $\int_{-2}^{2} sin\, x\, dx$ is equal to the area above the x-axis minus the area below the x-axis.




The red area cancels the blue area and $\int_{-2}^{2} sin\, x\, dx = 0$.