U5 Day 1

Tuesday, December 01, 2015
9:45 AM

Optimizatio
n Lesson D...

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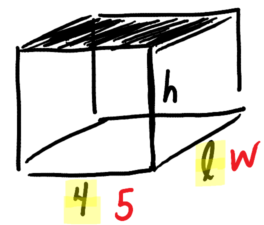
Unit 5 Optimization – Section 4.4

Definition: To *optimize*: to maximize / minimize

Example 1: "The Escalante Problem" 1982 AB #6 A tank with a rectangular base and rectangular sides is to be open at the top. It's constructed such that its width is 4 m and its volume is 36 m^3. To build the tank, it costs $10/sq meter for the base and $5/m^2 for the sides. What is the cost of the least expensive tank?

min cost



$$SA = 2(4h) + 2lh + 4l$$

\$5   \$5   \$10

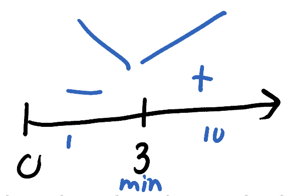
$$V = 36$$
$$4hl = 36$$
$$hl = 9$$
$$h = \frac{9}{l}$$

$$C = 40h + 10lh + 40l$$
$$C = 40\left(\frac{9}{l}\right) + 10l\left(\frac{9}{l}\right) + 40l$$
$$C = 360l^{-1} + 90 + 40l$$
$$C' = -360l^{-2} + 40$$
$$C' = \frac{-360}{l^2} + 40 = 0$$

$$\frac{-360}{l^2} = -40$$
$$-360 = -40l^2$$
$$9 = l^2$$
$$3 = l$$

$$C = 40(3) + 10(9) + 40(3)$$

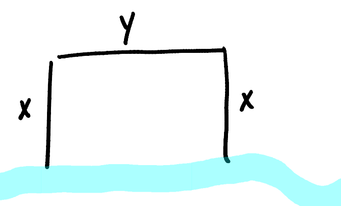
$$\boxed{\text{Cost} = \$330}$$



Example 2: A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

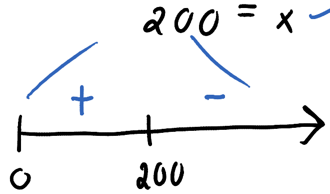
max area

$$P = 800\text{m}$$
$$800 = 2x + y$$
$$800 - 2x = y$$
$$800 - 400 = y$$



$$A = xy$$
$$A = x(800 - 2x)$$
$$A = 800x - 2x^2$$
$$\frac{dA}{dx} = 800 - 4x = 0$$
$$800 = 4x$$
$$200 = x$$

$$\boxed{\begin{array}{l}\text{Area} = 80{,}000 \text{ m}^2 \\ y = \text{Length} = 400 \text{ m} \\ x = \text{Width} = 200 \text{m}\end{array}}$$



MANAGERIAL CHALLENGE

A Skeleton in the Stealth Bomber's Closet[1]

In 1990 the U.S. Air Force publicly unveiled its newest long-range strategic bomber—the B-2 or "Stealth" bomber. This plane is characterized by a unique flying wing design engineered to evade detection by enemy radar. The plane has been controversial because of its high cost. However, a lesser-known controversy relates to its fundamental design.

The plane's flying wing design originated from a secret study of promising military technologies that was undertaken at the end of World War II. The group of prominent scientists who undertook the study concluded that a plane can achieve maximum range if it has a design in which virtually all the volume of the plane is contained in the wing. A complex mathematical appendix was attached to the study that purported to show that range could be *maximized* with the flying wing design.

However, a closer examination of the technical appendix by Joseph Foa, now an emeritus professor of engineering at George Washington University, discovered that a fundamental error had been made in the initial report. It turned out that the original researchers had taken the first derivative of a complex equation for the range of a plane and found that it had two solutions. The original researchers mistakenly concluded that the all-wing design was the one that maximized range, when, in fact, it *minimized* range.

In this chapter we introduce some of the same optimization techniques applied to an analysis of the Stealth bomber project. We develop tools designed to maximize profits or minimize costs. Fortunately, the mathematical functions we deal with in this chapter and throughout the book are much simpler than those that confronted the original "flying wing" engineers. We introduce techniques that can be used to check whether a function, such as profits or costs, is being minimized or maximized at a particular level of output.

[1] This Managerial Challenge is based primarily on W. Biddle, "Skeleton Alleged in the Stealth Bomber's Closet," *Science*, 12 May 1989, pp. 650–651.

Example 3: Suppose the product of two positive numbers is 64. What is the minimum sum of the square of each number?

min sum

$xy = 64$

$y = \frac{64}{x} \rightarrow y = \frac{64}{8} = 8$

$Sum = x^2 + y^2$

$S = x^2 + \left(\frac{64}{x}\right)^2 \rightarrow \frac{64^2}{x^2}$

$S = x^2 + 4096x^{-2}$

$S' = 2x - 8192x^{-3}$

$= 2x - \frac{8192}{x^3} = 0$

$2x = \frac{8192}{x^3}$

$2x^4 = 8192$

$x^4 = 4096$

$x = 8$

S': $-$ on $(0, 8)$, $+$ after 8

x: 0, 8

$$\boxed{Sum = 8^2 + 8^2 = 128}$$

Economics – Maximizing Profit/Minimizing Cost

$r(x)$ = the revenue from selling x items $\qquad \frac{dr}{dx}$ = marginal revenue

$c(x)$ = the cost of producing x items $\qquad \frac{dc}{dx}$ = marginal cost

$p(x) = r(x) - c(x)$ = the profit from selling x items $\qquad \frac{dp}{dx}$ = marginal profit

Maximum Profit: Occurs at the level at which marginal revenue = marginal cost

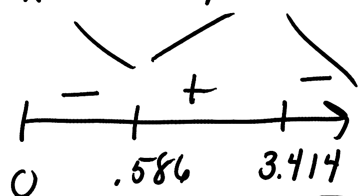
Why? $p'(x) = 0$

$$p(x) = r(x) - c(x)$$
$$p'(x) = r'(x) - c'(x) = 0$$
$$r'(x) = c'(x)$$

Example 4: Suppose that revenue is $r(x) = 9x$ and the cost is $c(x) = x^3 - 6x^2 + 15x$, where x represents thousands of units. Is there a production level that maximizes profit and if so, what is it?

$$p(x) = 9x - (x^3 - 6x^2 + 15x)$$
$$p(x) = 9x - x^3 + 6x^2 - 15x$$
$$p'(x) = 9 - 3x^2 + 12x - 15 = 0$$

$x \approx .586,\ 3.414$



$\boxed{3414 \text{ units}}$

Minimum Average Cost: Occurs at the level at which average cost = marginal cost

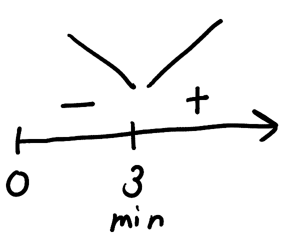
Why? $\text{Avg cost} = \frac{c(x)}{x}$

$$A' = \frac{x \cdot c'(x) - c(x) \cdot 1}{x^2} = 0 \rightarrow x\,c'(x) - c(x) = 0$$
$$x\,c'(x) = c(x)$$
$$c'(x) = \frac{c(x)}{x}$$

Example 5: Suppose $\boxed{c(x) = x^3 - 6x^2 + 15x}$ where x represents thousands of units. Is there a production level that minimizes average cost and if so, what is it?

$$\text{Avg cost} = \frac{c(x)}{x} = \frac{x^3 - 6x^2 + 15x}{x} = x^2 - 6x + 15$$

$$A' = 2x - 6 = 0$$
$$2x = 6$$
$$x = 3$$



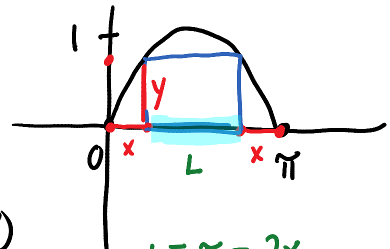
$\boxed{3000 \text{ units}}$

Example 6: A rectangle is to be inscribed under one arch of the sine curve. What is the largest area the rectangle can have, and what dimensions give that area?

$$Area = L \cdot W$$

$$A = (\pi - 2x)(\sin x)$$

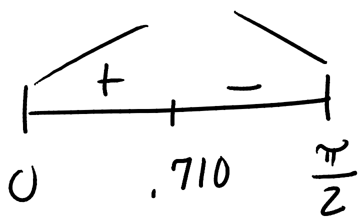
$$A' = (\pi - 2x)\cdot \cos x + \sin x \cdot -2 = 0$$



$L = \pi - 2x$

$W = y = \sin x$

$x \approx .710$



$$\boxed{\begin{aligned} \text{Length} &= 1.721 \text{ units} \\ \text{Width} &= .652 \text{ units} \\ \text{Area} &= 1.122 \text{ units} \end{aligned}}$$

Example 7: An open-top box is to be made by cutting congruent squares of side length x from the corners of a ~~20 by 25~~ 8.5 × 11 inch sheet of ~~film~~ paper and bending up the sides. How large should the squares be to make the box hold as much as possible? What is the resulting maximum volume?

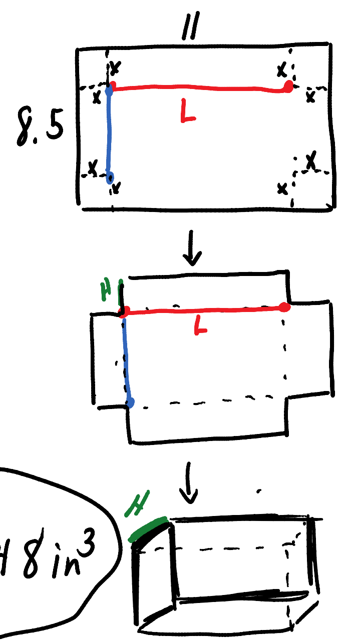
$(0, 4.25)$

$$V = L \times W \times H$$

$L = 11 - 2x$

$W = 8.5 - 2x$

$H = x$



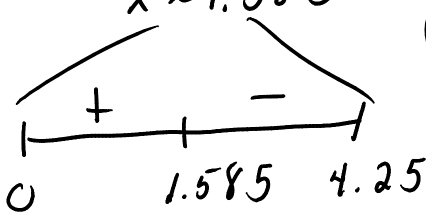
$$V = (11 - 2x)(8.5 - 2x)(x)$$

$$V = (93.5 - 39x + 4x^2)x$$

$$V = 93.5x - 39x^2 + 4x^3$$

$$V' = 93.5 - 78x + 12x^2 = 0$$

$x \approx 1.585$



$V \approx 66.148 \text{ in}^3$